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Half polarized $U(1)$ symmetric vacuum spacetimes with AVTD behavior

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Abstract

In a previous work, we used a polarization condition to show that there is a family of $U(1)$ symmetric solutions of the vacuum Einstein equations on $\Sigma \times S^1 \times R$ (Σ any two-dimensional manifold) such that each exhibits AVTD[1] behavior in the neighbourhood of its singularity. Here we consider the general case of S^1 bundles over the base $\Sigma \times R$ and determine a condition, called the half polarization condition, necessary and sufficient in our context, for AVTD behavior near the singularity.

Keywords: AVTD behavior; Einstein equations; Spacetime

1. Introduction

A rigorous study of the singularities in cosmological solutions of the vacuum Einstein equations has been hampered by the fact that the generic such solution is expected to have a singularity of the oscillatory type predicted by Belinsky, Lifshitz and Khalatnikov [BLK]. There is currently no satisfactory mathematical method for treating such oscillating singularities, at least when the spacetimes under study are spatially inhomogeneous. For this reason much effort has gone into the study of families of solutions which have milder

[1] Asymptotic velocity term dominated.
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cosmological singularities such as those of AVTD (asymptotically velocity term dominated) type, for which rigorous, so called Fuchsian, methods are available.

To suppress the oscillatory behavior expected for the generic solution, one can:

(i) introduce suitable matter sources such as scalar fields and study the solutions of the associated non vacuum field equations [7],
(ii) study higher dimensional models motivated by string or supergravity theories wherein (for sufficiently high dimensions at least) the oscillations are naturally suppressed [8], or
(iii) remain in $3+1$ dimensions but impose a combination of symmetry and polarization conditions in order to achieve the desired AVTD behavior.

For the case of $U(1)$ symmetric vacuum solutions on the trivial S^1 bundle $T^2 \times R \times S^1 \to T^2 \times R$ (with $U(1)$ symmetry imposed on the circular fibers) Isenberg and Moncrief [5] have showed, using Fuchsian methods, that AVTD behavior is achieved provided the solutions considered are at least half polarized in a certain well defined sense. The half polarization condition includes, as a special case, the fully polarized solutions wherein the 3 planes orthogonal to orbits of the $U(1)$ isometry action are integrable and the vacuum $3+1$ field equations reduce to a system of $2+1$ dimensional Einstein equations coupled to a massless scalar field on the quotient manifold $T^2 \times R$. The more general (half polarized) solutions admit, in addition, half the extra (asymptotic) Cauchy data expected for a fully general, non polarized solution of the same ($U(1)$ symmetric) type. On the basis of numerical studies due to Berger and Moncrief the fully general, non polarized $U(1)$ symmetric vacuum solution on $T^2 \times R \times S^1 \to T^2 \times R$ is expected to have an oscillatory singularity and hence not to be amenable to Fuchsian analysis [1].

Choquet-Bruhat, Isenberg and Moncrief have extended the analysis given in [5] to cover the case of polarized $U(1)$ symmetric vacuum solutions on manifolds of the more general type $\Sigma^2 \times R \times S^1 \to \Sigma^2 \times R$, where Σ^2 is an arbitrary compact surface and the bundle (in view of the assumed polarization condition) is necessarily trivial. In the present paper the polarization restriction is eliminated in favor of an appropriate half polarization condition and the limitation to trivial S^1 bundles over the base $\Sigma^2 \times R$ is also removed. The present work thus demonstrates the existence of a large family of vacuum $U(1)$ symmetric solutions of half polarized type defined on trivial and non trivial bundles over $\Sigma^2 \times R$ (with Σ^2 an arbitrary compact surface) and having AVTD singularity behavior. The half polarization condition used in [5] involved requiring one of the asymptotic functions to vanish. The half polarization condition which we find here necessary and sufficient for possible AVTD behavior can be understood in terms of the behavior of the VTD solutions to which our solutions converge as one approaches the singularity. Specifically a VTD solution is half polarized if and only if the set of geodesics in the Poincaré plane which represent it (at different spatial points) all tend to the same point as t approaches the singularity.

2. Einstein equations

A spacetime metric on a manifold $V_4 \equiv M \times R$, with M an S^1 principal fiber bundle over a surface Σ, reads, if it is invariant under the S^1 action on V_4,

$$^{(4)}g \equiv \mathrm{e}^{-2\phi(3)}g + \mathrm{e}^{2\phi}(\mathrm{d}\theta + A)^2 \tag{2.1}$$

with θ a parameter on the (spacelike) circular orbit, ϕ a scalar, A a locally defined one-form and $^{(3)}g$ a lorentzian metric, all on $V_3 := \Sigma \times R$.

The vacuum $3+1$ Einstein equations $Ricci(^{(4)}g) = 0$ for such an S^1 symmetric metric on V_4 are known [2,3] to be equivalent[2] to the wave map equation from $(V_3, {}^{(3)}g)$ into the Poincaré plane $P =: (R^2, G)$, $\Phi \equiv (\phi, \omega)$: $V_3 \to R^2$, where

$$G \equiv 2(\mathrm{d}\phi)^2 + \frac{1}{2}\mathrm{e}^{-4\gamma}(\mathrm{d}\omega)^2, \tag{2.2}$$

coupled to the $2+1$ Einstein equations for $^{(3)}g$ on V_3 with the wave map as the source field. The scalar function ω on V_3 is linked to the differential F of A by the relation

$$\mathrm{d}\omega = \mathrm{e}^{4\phi} * F, \quad \text{with } F = \mathrm{d}A. \tag{2.3}$$

Thus in local coordinates x^α, $\alpha = 0, 1, 2$, on V_3, with η the volume form of $^{(3)}g =^{(3)} g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta$, one has

$$F_{\alpha\beta} \equiv \frac{1}{2}\mathrm{e}^{-4\phi}\eta_{\alpha\beta\lambda}\partial^\lambda\omega. \tag{2.4}$$

The wave map equations are, with $^{(3)}\nabla$ the covariant derivative in the metric $^{(3)}g$

$$g^{\alpha\beta}\left({}^{(3)}\nabla^\alpha\partial_\beta\phi + \frac{1}{2}\mathrm{e}^{-4\phi}\partial_\alpha\omega\partial_\beta\omega\right) = 0, \tag{2.5}$$

$$g^{\alpha\beta}({}^{(3)}\nabla^\alpha\partial_\beta\omega - 4\partial_\alpha\omega\partial_\beta\phi) = 0. \tag{2.6}$$

The $2+1$ Einstein equations are, with "$\cdot$" indicating a scalar product in the metric G

$$^{(3)}R^{\alpha\beta} = \partial_\alpha\Phi \cdot \partial_\beta\Phi. \tag{2.7}$$

To solve these equations we choose for $^{(3)}g$ a zero shift, we denote the lapse by e^λ and we weigh by e^λ, without restricting the generality, the t dependent space metric $g = g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b$, $a, b = 1, 2$. That is, we set

$$^{(3)}g \equiv -N^2\,\mathrm{d}t^2 + g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b \quad \text{with } N \equiv \mathrm{e}^\lambda,\ g_{ab} \equiv \mathrm{e}^\lambda\sigma_{ab}. \tag{2.8}$$

We denote by σ^{ab} the contravariant form of σ. The extrinsic curvature of Σ_t in $(V_3,{}^{(3)}g)$ is

$$k_{ab} := -\frac{1}{2N}\partial_t g_{ab} \equiv -\frac{1}{2}(\sigma_{ab}\partial_t\lambda + \partial_t\sigma_{ab}). \tag{2.9}$$

The mean extrinsic curvature τ is therefore

$$\tau := g^{ab}k_{ab} \equiv -\mathrm{e}^{-\lambda}\left(\partial_t\lambda + \frac{1}{2}\psi\right), \tag{2.10}$$

where we have defined

$$\psi := \sigma^{ab}\partial_t\sigma_{ab}. \tag{2.11}$$

[2] If we choose an arbitrary harmonic one-form appearing in the solution to be zero.

The connection coefficients (Christoffel symbols) of ${}^{(3)}g$ are found to be (note that ${}^{(3)}g^{00} = -\mathrm{e}^{-2\lambda}, {}^{(3)}g^{ab} = g^{ab} = \mathrm{e}^{-\lambda}\sigma^{ab}$)

$$ {}^{(3)}\Gamma_c^{ab} = \Gamma_{ab}^c(g) = \Gamma_{ab}^c(\sigma) + \frac{1}{2}(\delta_b^c\partial_a\lambda + \delta_a^c\partial_b\lambda - \sigma^{cd}\sigma_{ab}\partial_d\lambda), \tag{2.12} $$

$$ {}^{(3)}\Gamma_0^{00} = \partial_t\lambda \quad {}^{(3)}\Gamma_0^{0a} = \partial_a\lambda, \quad {}^{(3)}\Gamma_a^{00} = \sigma^{ab}\,\mathrm{e}^{\lambda}\,\partial_a\lambda, \tag{2.13} $$

$$ {}^{(3)}\Gamma_0^{ab} = -\mathrm{e}^{-\lambda}k_{ab}, \qquad {}^{(3)}\Gamma_b^{a0} = -\mathrm{e}^{\lambda}k_a^b. \tag{2.14} $$

In particular it holds that

$$ {}^{(3)}g^{\alpha\beta(3)}\Gamma_{\alpha\beta}^0 = \frac{1}{2}\psi\,\mathrm{e}^{-2\lambda}. \tag{2.15} $$

We see that the metric ${}^{(3)}g$ is in harmonic time gauge if and only if $\psi = 0$.

The Einstein equations split into constraints and evolution equations. We denote by $S_\beta^\alpha \equiv {}^{(3)}R_\beta^\alpha - \frac{1}{2}\delta_\beta^{\alpha(3)}R$ the Einstein tensor of ${}^{(3)}g$, by T_β^α the stress energy tensor of Φ, and we set $\Sigma_\beta^\alpha \equiv S_\beta^\alpha - T_\beta^\alpha$. The constraints are:

$$ C_0 \equiv \Sigma_0^0 \equiv -\frac{1}{2}\{R(g) - k\cdot k + \tau^2 - \mathrm{e}^{-2\lambda}\partial_t\Phi\cdot\partial_t\Phi - g^{ab}\partial_a\Phi\cdot\partial_b\Phi\} = 0 \tag{2.16} $$

and (indices raised with g^{ab}, ∇ the covariant derivative in the metric g)

$$ C_a \equiv \mathrm{e}^{\lambda}\,\Sigma_a^0 \equiv -\{\nabla_b k_a^b - \partial_a\tau + \mathrm{e}^{-\lambda}\partial_a\Phi\cdot\partial_t\Phi\} = 0. \tag{2.17} $$

The evolution equations are, with $N = \mathrm{e}^{\lambda}$,

$$ N({}^{(3)}R_a^b - \rho_a^b) \equiv -\partial_t k_a^b + N\tau k_a^b - \nabla^b\partial_a N + NR_a^b - N\partial_a\Phi\cdot\partial^b\Phi = 0. \tag{2.18} $$

In order to obtain a first order system in the Fuchsian analysis that we will make, we introduce auxiliary unknowns Φ_t, Φ_a, σ_c^{ab} which are identified with the first partial derivatives of Φ and the covariant derivative of σ with respect to a given t independent metric $\tilde{\sigma}$. These new unknowns satisfy the evolution equations

$$ \partial_t\Phi = \Phi_t, \tag{2.19} $$

$$ \partial_t\Phi_a = \partial_a\Phi_t, \tag{2.20} $$

$$ \partial_t\sigma_c^{ab} = \tilde{\nabla}_c\partial_t\sigma^{ab}, \tag{2.21} $$

where, by the definitions of σ and k,

$$ \partial_t\sigma^{ab} = 2\,\mathrm{e}^{2\lambda}k^{ab} + \sigma^{ab}\partial_t\lambda. \tag{2.22} $$

The function λ is not left unknown, but rather is determined by a gauge condition from its VTD value.

3. VTD equations and solutions

The Velocity Terms Dominated equations are obtained by dropping the space derivatives in the equations.

We denote by a tilde quantities which are independent of t, and we denote VTD solutions using a hat.

3.1. Einstein evolution VTD solutions

In order to obtain a global (on Σ) formulation we choose a VTD metric which remains in a fixed conformal class over Σ as t evolves; we set

$$\hat{\sigma}_{ab} = \tilde{\sigma}_{ab} \quad \text{and} \quad \hat{g}_{ab} = \mathrm{e}^{\hat{\lambda}} \tilde{\sigma}_{ab}. \tag{3.1}$$

Then

$$\hat{\psi} = 0, \quad \hat{g}^{ab} = \mathrm{e}^{-\hat{\lambda}} \tilde{\sigma}^{ab}, \quad \partial_t \hat{g}_{ab} = \mathrm{e}^{\hat{\lambda}} \tilde{\sigma}_{ab} \partial_t \hat{\lambda}, \tag{3.2}$$

and the definition of k gives that

$$\hat{k}_{ab} = -\frac{1}{2} \tilde{\sigma}_{ab} \partial_t \hat{\lambda}, \quad \hat{k}_a^b = -\frac{1}{2} \mathrm{e}^{-\hat{\lambda}} \delta_a^b \partial_t \hat{\lambda}, \quad \hat{\tau} := \hat{k}_a^a = -\mathrm{e}^{-\hat{\lambda}} \partial_t \hat{\lambda}. \tag{3.3}$$

Requiring that these VTD quantities satisfy the VTD evolution equations, we obtain

$$\partial_t \hat{k}_a^b = \hat{N} \hat{\tau} \hat{k}_a^b. \tag{3.4}$$

Therefore, by straightforward computation

$$\partial_{tt}^2 \hat{\lambda} = 0; \quad \text{hence } \hat{\lambda} = \tilde{\lambda} - \tilde{v} t \tag{3.5}$$

with $\tilde{\lambda}$ and $\tilde{v}$ arbitrary functions on Σ, independent of t. Then we have

$$\hat{k}_{ab} = \frac{1}{2} \tilde{v} \tilde{\sigma}_{ab}, \quad \hat{k}_a^b = \frac{1}{2} \tilde{v}\, \mathrm{e}^{-\hat{\lambda}} \delta_a^b, \quad \hat{\tau} = \mathrm{e}^{-\hat{\lambda}} \tilde{v}. \tag{3.6}$$

3.2. Wave map VTD solutions

The results for a VTD wave map are very different from the results obtained for a scalar function [4]. If we drop space derivatives in the wave map equations we obtain geodesic equations in the target manifold, with t the length parameter on these geodesics so long as the $2+1$ metric is in harmonic time gauge. If we make the change of coordinates $Y = \mathrm{e}^{2\phi}$ in the target (which defines a diffeomorphism from R^2 onto the upper half plane $Y > 0$), the metric G takes a standard form for the metric of a Poincaré half plane; namely

$$G \equiv \frac{1}{2} \left\{ \frac{\mathrm{d}\omega^2 + \mathrm{d}Y^2}{Y^2} \right\}, \quad Y = \mathrm{e}^{2\phi}. \tag{3.7}$$

The VTD, geodesic, equations written in this metric read, with a prime denoting the derivative with respect to t

$$\omega'' - 2Y^{-1}\omega' Y' = 0, \tag{3.8}$$

$$Y'' + Y^{-1}\omega' X' = 0. \tag{3.9}$$

The general solution of these geodesic equations is represented in these coordinates, as is well known, by half circles[3] centered on the line $Y = 0$; specifically, with A and B arbitrary constants (that is, independent of t), the solution takes the form

$$\hat{\omega} = B + A\cos\theta, \qquad \hat{Y} = A\sin\theta, \quad 0 < \theta < \pi. \tag{3.10}$$

These functions ω and Y satisfy the differential equations (3.8), (3.9) if and only if it holds that:

$$\frac{\theta''}{\theta'} = \frac{\cos\theta}{\sin\theta}\theta'. \tag{3.11}$$

Integrating this equation we have that, with $\tilde{w}$ independent of t

$$\theta' = -\tilde{w}\sin\theta. \tag{3.12}$$

Another integration gives that, with $\tilde{\Theta}$ independent of t

$$\tan\frac{\theta}{2} = \tilde{\Theta}\,\mathrm{e}^{-\tilde{w}t}. \tag{3.13}$$

If we now make the substitution $A = \mathrm{e}^{2\tilde{\phi}}$ and $B = \tilde{\omega}$, then (3.10) reads

$$\hat{\phi} = \tilde{\phi} + \frac{1}{2}\log(\sin\theta), \qquad \hat{\omega} = \tilde{\omega} + \mathrm{e}^{2\tilde{\phi}}\cos\theta. \tag{3.14}$$

Remark 3.1. The set of above formulas is identical to the following one

$$\hat{\omega} \equiv \tilde{\omega} + \mathrm{e}^{2\tilde{\phi}}\frac{1 - \tilde{\Theta}^2\,\mathrm{e}^{-2\tilde{w}t}}{1 + \tilde{\Theta}^2\,\mathrm{e}^{-2\tilde{w}t}}, \qquad \mathrm{e}^{2\hat{\phi}} \equiv \hat{Y} = \mathrm{e}^{2\tilde{\phi}}\frac{2\tilde{\Theta}\,\mathrm{e}^{-\tilde{w}t}}{1 + \tilde{\Theta}^2\,\mathrm{e}^{-2\tilde{w}t}}. \tag{3.15}$$

$\hat{Y}$ tends to zero when t tends to ∞, but $\hat{\omega}$ tends to $\tilde{\omega} + \mathrm{e}^{2\tilde{\phi}}$.

3.3. Einstein constraint VTD solutions.

We deduce from (3.14) and (3.12) that

$$\partial_t\hat{\phi} \equiv \frac{1}{2}\hat{Y}^{-1}\hat{Y}' = \frac{1}{2}\frac{\cos\theta\theta'}{\sin\theta} = -\frac{1}{2}\tilde{w}\cos\theta \tag{3.16}$$

and

$$\mathrm{e}^{-2\hat{\phi}}\partial_t\hat{\omega} = -\theta' = \tilde{w}\sin\theta. \tag{3.17}$$

The Einstein VTD constraints reduce to the following equation:

$$2\hat{C}_0 \equiv -\hat{k}\cdot\hat{k} + \hat{\tau}^2 - \mathrm{e}^{-2\hat{\lambda}}\left\{2(\partial_t\hat{\phi})^2 + \frac{1}{2}\mathrm{e}^{-4\hat{\phi}}(\partial_t\hat{\omega})^2\right\} = 0. \tag{3.18}$$

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[3] We discard here the special case which corresponds to the polarized case, treated elsewhere, where these circles are centered at infinity. The geodesics are then the half lines $X \equiv \omega = \text{constant}$.

We have, using (3.16) and (3.17),

$$2(\partial_t\hat{\phi})^2 + \frac{1}{2}\mathrm{e}^{-4\hat{\phi}}(\partial_t\hat{\omega})^2 = \frac{1}{2}\tilde{w}^2. \tag{3.19}$$

We deduce therefore from (3.6) that the VTD constraint (3.18) is satisfied if and only if

$$\tilde{v}^2 = \tilde{w}^2. \tag{3.20}$$

4. Fuchsian expansion

4.1. $2+1$ metric expansions

For the unknowns σ and k we choose the following expansions, with the various $\varepsilon' s$ being positive numbers to be chosen later

$$\sigma^{ab} = \tilde{\sigma}^{ab} + \mathrm{e}^{-\varepsilon_\sigma t} u_\sigma^{ab}, \tag{4.1}$$

$$k_a^b = \mathrm{e}^{-\lambda}\left(\frac{1}{2}\tilde{v}\delta_a^b + \mathrm{e}^{-\varepsilon_k t} u_{k,a}^b\right). \tag{4.2}$$

Then

$$\tau \equiv k_a^a = \mathrm{e}^{-\lambda}(\tilde{v} + \mathrm{e}^{-\varepsilon_k t} u_{k,a}^a). \tag{4.3}$$

We take as a gauge condition

$$\lambda = \hat{\lambda}; \quad \text{hence } \partial_t\lambda = -\tilde{v}. \tag{4.4}$$

Comparing the expressions (4.4) and (2.10) for τ, we find that this condition is equivalent to the gauge fixing requirement

$$\mathrm{e}^{-\varepsilon_k t} u_{k,a}^a + \frac{1}{2}\psi = 0. \tag{4.5}$$

4.2. Wave map expansion

We expand Φ near its VTD value; that is we set

$$\phi = \hat{\phi} + \mathrm{e}^{-\varepsilon_\phi t} u_\phi \quad \text{with } \hat{\phi} = \tilde{\phi} + \frac{1}{2}\log(\sin\theta), \tag{4.6}$$

while for ω, for convenience of computation, we choose to set

$$\omega = \hat{\omega} + \mathrm{e}^{2\phi}\,\mathrm{e}^{-\varepsilon_\omega t} u_\omega \quad \text{with } \hat{\omega} = \tilde{\omega} + \mathrm{e}^{2\tilde{\phi}}\cos\theta. \tag{4.7}$$

4.3. Expansion for first derivatives

We expand the auxiliary unknowns near the values of the derivatives of the VTD solution. That is we set (see (3.16) and (3.17))

$$\phi_t = \partial_t\hat{\phi} + \mathrm{e}^{-\varepsilon_{\phi_t} t} u_{\phi_t} \equiv -\frac{1}{2}\tilde{w}\cos\theta + \mathrm{e}^{-\varepsilon_{\phi_t} t} u_{\phi_t}, \tag{4.8}$$

$$\omega_t = \partial_t\hat{\omega} + e^{2\phi}\,e^{-\varepsilon_{\omega_t}}u_{\omega_t} \equiv e^{2\tilde{\phi}}\tilde{w}\,\sin^2\theta + e^{2\phi}\,e^{-\varepsilon_{\omega_t}}u_{\omega_t}. \tag{4.9}$$

The expansions of ϕ_a and ω_a are defined similarly by setting

$$\phi_a = \partial_a\hat{\phi} + e^{-\varepsilon_{\phi'}}u_{\phi_a}, \qquad \omega_a = \partial_a\hat{\omega} + e^{2\phi}\,e^{-\varepsilon_{\omega'}t}u_{\omega_a}. \tag{4.10}$$

We next compute $\partial_a\hat{\phi}$ and $\partial_a\hat{\omega}$. It follows from (3.14) that

$$\partial_a\hat{\phi} = \partial_a\tilde{\phi} + \frac{\cos\,\theta}{2\sin\,\theta}\partial_a\theta, \qquad \partial_a\hat{\omega} = \partial_a\tilde{\omega} + e^{2\tilde{\phi}}(2\cos\,\theta\partial_a\tilde{\phi} - \sin\,\theta\partial_a\theta). \tag{4.11}$$

We compute $\partial_a\theta$ using (3.13) and elementary properties of sine and cosine. We find that

$$\partial_a\theta = \tilde{\Theta}^{-1}\sin\,\theta\partial_a(\tilde{\Theta} - t\tilde{w}). \tag{4.12}$$

Therefore it holds that

$$\partial_a\hat{\phi} = \partial_a\tilde{\phi} + \tilde{\Theta}^{-1}\frac{\cos\theta}{2}\partial_a(\tilde{\Theta} - \tilde{w}t). \tag{4.13}$$

Then, writing $\partial_a\hat{\omega}$ as sum of a term independent of t plus terms tending to zero when t tends to infinity, we have

$$\partial_a\hat{\omega} \equiv \partial_a(\tilde{\omega} + e^{2\tilde{\phi}}) - e^{2\tilde{\phi}}[2(1-\cos\,\theta)\partial_a\tilde{\phi} - \tilde{\Theta}^{-1}\sin^2\theta\partial_a(\tilde{\Theta} - \tilde{w}t)]. \tag{4.14}$$

For σ_c^{ab}, since $\tilde{\nabla}_c\tilde{\sigma}^{ab} = 0$, we set

$$\sigma_c^{ab} \equiv e^{-\varepsilon_{\sigma'}t}u_{\sigma',c}^{ab}. \tag{4.15}$$

5. Fuchsian system for the evolution equations

Given the Fuchsian expansions of the previous section, the Einstein-wave map evolution system reads as a first order system for the set of unknowns $U \equiv (u_\sigma, u_k, u_\Phi, u_{\Phi_t}, u_{\Phi'}, u_{\sigma'})$.

The differential system for U is Fuchsian in a neighbourhood of $t = +\infty$ if it takes the form

$$\partial_t U - LU = e^{-\mu t}F(t, x, U, \tilde{\partial}U) \tag{5.1}$$

with L a linear operator independent of t with non negative eigenvalues, μ a positive number and F a set of tensor fields linear in $\tilde{\partial}U$, continuous in t, analytic in x and U and uniformly Lipshitzian in all its arguments in a neighbourhood of $U = 0$, for t large enough.

5.1. Einstein evolution equations

5.1.1. Equation for u_σ

The Fuchsian expansion (4.2) for k yields the following equation:

$$\partial_t g^{ab} \equiv 2Nk^{ab} \equiv 2\,e^{\lambda}g^{ac}k_c^b \equiv e^{-\lambda}(\tilde{v}\sigma^{ab} + 2\,e^{-\varepsilon_k t}\sigma^{ac}u_{k,c}^b). \tag{5.2}$$

Using $g^{ab} \equiv e^{-\lambda}\sigma^{ab}$ and $\partial_t\lambda = -\tilde{v}$, we have

$$\partial_t g^{ab} \equiv e^{-\lambda}(\tilde{v}\sigma^{ab} + \partial_t\sigma^{ab}). \tag{5.3}$$

Combining these equations together with the Fuchsian expansion of σ results in the equation:

$$\partial_t u_\sigma^{ab} - \varepsilon_\sigma u_\sigma^{ab} = 2\,\mathrm{e}^{(\varepsilon_\sigma - \varepsilon_k)t}\sigma^{ac}u_{k,c}^b, \tag{5.4}$$

which is of Fuchsian type if $\varepsilon_k > \varepsilon_\sigma > 0$.

5.1.2. Equation for u_k

The Fuchsian expansion of k together with $N = \mathrm{e}^\lambda$ and $\partial_t\lambda = -\tilde{v}$ imply by straightforward computation that

$$\partial_t k_a^b \equiv \mathrm{e}^{-\lambda}\left\{\frac{1}{2}\tilde{v}^2\delta_a^b + \mathrm{e}^{-\varepsilon_k t}(\tilde{v} - \varepsilon_k)u_{k,a}^b + \mathrm{e}^{-\varepsilon_k t}\partial_t u_{k,a}^b\right\}, \tag{5.5}$$

and

$$N\tau k_a^b \equiv \mathrm{e}^{-\lambda}\{(\tfrac{1}{2}\tilde{v}^2\delta_a^b + \tilde{v}\,\mathrm{e}^{-\varepsilon_k t}u_{k,a}^b) + \mathrm{e}^{-\varepsilon_k t}u_{k,c}^c(\tfrac{1}{2}\tilde{v}\delta_a^b + \mathrm{e}^{-\varepsilon_k t}u_{k,a}^b). \tag{5.6}$$

We see that $\mathrm{e}^{-\lambda}\tilde{v}^2$ disappears from the difference $\partial_t k_a^b - N\tau k_a^b$, which motivates the choice of the Fuchsian expansion.

To write the evolution Eq. (2.18) for k we now compute

$$\nabla^b\partial_a N \equiv \mathrm{e}^{-\lambda}\sigma^{bc}\nabla_c\partial_a\,\mathrm{e}^\lambda \equiv \sigma^{bc}[\partial_c\lambda\partial_a\lambda + \partial_a\partial_c\lambda - \Gamma_{ac}^d(g)\partial_d\lambda]. \tag{5.7}$$

On the other hand, since Σ is two-dimensional and g is conformal to σ with a factor e^λ, we have that

$$NR_a^b \equiv \mathrm{e}^\lambda R_a^b \equiv \frac{1}{2}\mathrm{e}^\lambda\delta_a^b R(g) = \frac{1}{2}\delta_a^b\{R(\sigma) - \Delta_\sigma\lambda\}. \tag{5.8}$$

>From these results, if we define

$$f_a^b(t, u, u_x) := -\nabla^b\partial_a N + NR_a^b - N\partial_a\Phi\cdot\partial^b\Phi$$

then we calculate

$$f_a^b \equiv \sigma^{bc}[\partial_c\lambda\partial_a\lambda + \partial_a\partial_c\lambda - \Gamma_{ac}^d(g)\partial_d\lambda] + \frac{1}{2}\delta_a^b[R(\sigma) - \Delta_\sigma\lambda] - \sigma^{bc}\Phi_a\cdot\Phi_c. \tag{5.9}$$

We see that f_a^b is at most a second order polynomial in t, is analytic in x when $\tilde{v}$, $\tilde{w}$, $\tilde{\lambda}$, $\tilde{\sigma}$ are analytic; is linear in ∂u; and is analytic, bounded and Lipshitzian in u for u bounded and for large[4] t, except eventually for the last term which reads

$$\sigma^{bc}\Phi_a\cdot\Phi_c = 2\sigma^{bc}\left(\phi_a\phi_c + \frac{1}{2}\mathrm{e}^{-4\phi}\omega_a\omega_c\right). \tag{5.10}$$

The expansion (4.10) of ϕ_a shows that it does not cause problems for the boundedness of f_a^b. However the expansion of ω_a gives

$$\begin{aligned}\mathrm{e}^{-2\phi}\omega_a = \mathrm{e}^{-2\phi}\partial_a(\tilde{\omega} + \mathrm{e}^{2\tilde{\phi}}) - \mathrm{e}^{-2\phi+2\tilde{\phi}}[2(1-\cos\,\theta)\partial_a\tilde{\phi} \\ + \tilde{\Theta}^{-1}\sin^2\theta\partial_a(\tilde{\Theta} - \tilde{w}t)] + \mathrm{e}^{-\varepsilon_{\omega'}t}u_{\omega_a}.\end{aligned} \tag{5.11}$$

[4] This restriction on t comes from the covariant components of σ which remain bounded as long as σ^{ab} remains positive definite.

It follows from (4.6) that

$$e^{2(\tilde{\phi}-\phi)} = \frac{e^{-2\delta\phi}}{\sin\,\theta} \quad \text{with } \delta\phi \equiv e^{-\varepsilon_\phi t} u_\phi.$$

Therefore, using $(1-\cos\,\theta)/\sin\,\theta = \tan(\theta/2)$ we have:

$$e^{-2\phi}\omega_a = e^{2\tilde{\phi}} \frac{e^{-2\delta\phi}}{\sin\,\theta} \partial_a(\tilde{\omega} + e^{2\tilde{\phi}}) - e^{-2\delta\phi} \times \left[2\tan\frac{\theta}{2}\partial_a\tilde{\phi} + \tilde{\Theta}^{-1}\sin\,\theta\partial_a(\tilde{\Theta} - \tilde{\omega}t)\right] + e^{-\varepsilon_{\omega'}t} u_{\omega_a}. \tag{5.12}$$

We see that $e^{-2\phi}\omega_a$ will increase like $(\sin\,\theta)^{-1}$ - that is, like $e^{\tilde{\omega}t}$ - as t tends to infinity, except if

$$\tilde{\omega} + e^{2\tilde{\phi}} = \text{constant}. \tag{5.13}$$

Condition (5.13) is a generalization of the condition imposed on the fields in [5], with other notations, to obtain AVTD behavior, in the case that Σ is a torus. Following the terminology of [5] we call Eq. (5.13) the “half polarization” condition. Its geometric meaning is that the set of geodesics in the Poincaré plane representing the VTD solution all tend to the same point of the axis $Y = 0$ as t tends to infinity.

After inserting the Fuchsian expansions and multiplying by $e^{\lambda+\varepsilon_k:t}$ we find that Eq. (2.18) takes the form

$$\partial_t u^b_{k,a} - \varepsilon_k u^b_{k,a} - \frac{1}{2}\upsilon\delta^b_a u^c_{k,c} = e^{-\varepsilon_k t} u^c_{k,c} u^b_{k,a} + e^{\lambda+\varepsilon_k t} f^b_a(t, u, u_x). \tag{5.14}$$

Since $\lambda = \tilde{\lambda} - \tilde{\upsilon}t$ and $\tilde{\upsilon} = \tilde{\omega}$ this system can take a Fuchsian form only if the functions $\tilde{\omega}$ and $\tilde{\phi}$ satisfy the half polarization condition (5.13) . To obtain the system in obviously Fuchsian form in that case, we split (5.14) into its trace and its traceless parts. For the trace part we have

$$\partial_t u^a_{k,a} - \varepsilon_k u^a_{k,a} - \tilde{\upsilon} u^a_{k,a} = e^{-\varepsilon_k t} u^c_{k,c} u^a_{k,a} + e^{\lambda+\varepsilon_k t} f^a_a(t, u, u_x). \tag{5.15}$$

This equation takes Fuchsian form if and only if (5.13) is satisfied and $\tilde{\upsilon} > \varepsilon_k$. The same is verified for the traceless part ${}^T u^b_{k,a}$, which satisfies an equation with left hand side

$$\partial^T_t u^{k,a}_b - \varepsilon^T_k u^{k,a}_b. \tag{5.16}$$

5.1.3. *Equation for $u_{\sigma'}$*

Using the expansion of k and the relation $\partial_t\lambda = -\tilde{\upsilon}$, we find that

$$\partial_t \sigma^{ab} = 2\,e^{2\lambda} k^{ab} + \sigma^{ab}\partial_t\lambda = 2\,e^{-\varepsilon_k t}\sigma^{ac} u^b_{k,c}. \tag{5.17}$$

The equation for σ^{ab}_c gives therefore the following equation for $u_{\sigma'}$:

$$\partial_t u^{ab}_{\sigma',c} - \varepsilon_{\sigma'} u^{ab}_{\sigma',c} = 2\,e^{(\varepsilon_{\sigma'}-\varepsilon_k)t}\tilde{\nabla}_c[\sigma^{ac} u^b_{k,c}], \tag{5.18}$$

which is of Fuchsian type so long as $\varepsilon_{\sigma'} < \varepsilon_k$.

5.2. Wave map equations

5.2.1. Equations for auxiliary variables

The equations resulting from the introduction of the new variables ϕ_t, ω_t are

$$\partial_t\phi - \phi_t = 0, \qquad \partial_t\omega - \omega_t = 0. \tag{5.19}$$

The first equation is of Fuchsian type for u_ϕ if $\varepsilon_{\Phi_t} > \varepsilon_\Phi$, since it reads

$$\partial_t u_\phi - \varepsilon_\phi u_\phi = \mathrm{e}^{(-\varepsilon_{\Phi_t}+\varepsilon_\Phi)t} u_{\phi_t}. \tag{5.20}$$

The second equation reads

$$[\partial_t u_\omega + (2\phi_t - \varepsilon_\omega)u_\omega] - \mathrm{e}^{(\varepsilon_\Phi - \varepsilon_{\Phi_t})t} u_{\omega_t} = 0. \tag{5.21}$$

We replace ϕ_t by its value given in (4.8), which we write as follows:

$$\phi_t = -\frac{1}{2}\tilde{w} + \frac{1}{2}\tilde{w}(1-\cos\,\theta) + \mathrm{e}^{-\varepsilon_{\Phi_t}t}u_{\phi_t}. \tag{5.22}$$

Since $1-\cos\,\theta$ falls off to zero as $\mathrm{e}^{-2\tilde{w}t}$, Eq. (5.21) is of Fuchsian type for u_ω if $\tilde{w} > 0$ and $\varepsilon_{\Phi_t} > \varepsilon_\Phi$.

In Eqs. (2.20) to be satisfied by ϕ_a and ω_a, the derivatives of the VTD terms disappear, due to the commutation of partial derivatives. The equation for ϕ_a reads

$$\partial_t u_{\phi_a} - \varepsilon_{\phi'} u_{\phi_a} = \mathrm{e}^{-(\varepsilon_{\Phi_t}-\varepsilon_{\Phi'})t}(\partial_a u_{\phi_t} - t\partial_a w), \tag{5.23}$$

while the equation for ω_a becomes, using the expressions for ω_t and ω_a

$$\partial_t u_{\omega_a} + (2\phi_t - \varepsilon_{\omega'})u_{\omega_a} = \mathrm{e}^{-(\varepsilon_{\Phi_t}-\varepsilon_{\Phi'})t}(\partial_a u_{\omega_t} + 2\phi_a u_{\omega_t}). \tag{5.24}$$

These equations are of Fuchsian type so long as $\tilde{w} > 0$ and $\varepsilon_{\Phi_t} > \varepsilon_{\Phi'}$.

5.2.2. Equation for u_{Φ_t}

The first equation, (2.5), for the wave map reads

$$\begin{aligned} &g^{\alpha\beta}\left(\nabla_\alpha\partial_\beta\phi + \frac{1}{2}\mathrm{e}^{-4\phi}\partial_\alpha\omega\partial_\beta\omega\right) \\ &\quad\equiv -\mathrm{e}^{-2\lambda}\left(\partial_t\phi_t + \frac{1}{2}\mathrm{e}^{-4\phi}\omega_t\omega_t\right) \\ &\qquad + \mathrm{e}^{-\lambda}\sigma^{ab}\left(\nabla_a\phi_b + \frac{1}{2}\mathrm{e}^{-4\phi}\omega_a\omega_b\right) + g^{\alpha\beta}\Gamma^0_{\alpha\beta}\phi_t = 0. \end{aligned}$$

Using the Fuchsian expansions for ϕ_t and ω_t together with $\theta u = -\tilde{w}\,\sin\,\theta$ and the value given in Section 5.1.2 for $\mathrm{e}^{2(\tilde{\phi}-\phi)}$ we find that:

$$\begin{aligned} \partial_t\phi_t + \frac{1}{2}\mathrm{e}^{-4\phi}\omega_t\omega_t &\equiv \mathrm{e}^{-\varepsilon_{\Phi_t}t}(\partial_t u_{\phi_t} - \varepsilon_{\phi_t}u_{\phi_t}) - \frac{1}{2}\tilde{w}^2\sin^2\theta \\ &\quad + \frac{1}{2}(\mathrm{e}^{-2\delta\phi}\tilde{w}\,\sin\,\theta + \mathrm{e}^{-\varepsilon_{\omega_t}t}u_{\omega_t})^2. \end{aligned}$$

On the other hand, using the expansions for σ^{ab}, ϕ_a and ω_a we find that:

$$\mathrm{e}^{-\lambda}\sigma^{ab}\left(\nabla_a\phi_b+\frac{1}{2}\mathrm{e}^{-4\phi}\omega_a\omega_b\right)$$
$$\equiv \mathrm{e}^{-\lambda}(\tilde{\sigma}^{ab}+\delta\sigma^{ab})$$
$$\times\left\{\nabla_b\partial_a\hat{\phi}+\mathrm{e}^{-\varepsilon_{\Phi'}t}\nabla_b u_{\phi_a}+\frac{1}{2}(\mathrm{e}^{-2\phi}\partial_a\hat{\omega}+\mathrm{e}^{-\varepsilon_{\omega u}t}u_{\omega_a})(\mathrm{e}^{-2\phi}\partial_b\hat{\omega}+\mathrm{e}^{-\varepsilon_{\omega'}t}u_{\omega_b})\right\}.$$

We recall that

$$\nabla_b\partial_a\hat{\phi}\equiv\nabla_b\left[\partial_a\tilde{\phi}+\frac{\cos\theta}{2}\tilde{\Theta}^{-1}\partial_a(\tilde{\Theta}-\tilde{w}t)\right],$$

while under the half polarization assumption

$$\tilde{\omega}+\mathrm{e}^{2\tilde{\phi}}=\text{constant}, \tag{5.25}$$

the product $\mathrm{e}^{-2\phi}\partial_a\hat{\omega}$ is given by

$$\mathrm{e}^{-2\phi}\partial_a\hat{\omega}=-\mathrm{e}^{-2\delta\phi}\tilde{\Theta}^{-1}\sin\theta\partial_a(\tilde{\Theta}-\tilde{w}).$$

Finally we calculate

$$g^{\alpha\beta}\Gamma^0_{\alpha\beta}\equiv\frac{1}{2}\psi\,\mathrm{e}^{-2\lambda}=-\mathrm{e}^{-2\lambda-\varepsilon_k t}u^a_{k,a}. \tag{5.26}$$

Inserting these computations into the first wave map equation produces an equation of the form

$$\partial_t u_{\phi_t}-\varepsilon_{\Phi_t}u_{\phi_t}=\mathrm{e}^{-\mu t}f_{\phi_t}(x,t,u,\partial u) \tag{5.27}$$

which is of the Fuchsian type (5.1) (with $\mu>0$) so long as $\tilde{v}>\varepsilon_{\Phi_t}$, and $\varepsilon_k>\varepsilon_{\Phi_t}$.

Analogous computations show that the equation for u_{ω_t} is Fuchsian presuming these same inequalities hold.

5.3. Results for evolution

As a consequence of the calculations above we have proven the following theorem.

Theorem 5.1. *There exist a collection of positive numbers* $\{\varepsilon_\sigma,\varepsilon_{\sigma u},\varepsilon_k,\varepsilon_\Phi,\varepsilon_{\Phi_t},\varepsilon_{\Phi u}\}$ *such that, given analytic asymptotic data on* Σ, $\tilde{A}=\{\tilde{v}=\tilde{w},\tilde{\lambda},\tilde{\sigma},\tilde{\Theta},\tilde{\phi},\tilde{\omega}\}$, *the Einstein-wave map evolution system written in first order form for the unknown* U, *which defines* g, k, Φ *and auxiliary variables by the Fuchsian expansions of Section 4, is a Fuchsian system for* U *if and only if* $\tilde{\phi}$ *and* $\tilde{\omega}$ *satisfy the half polarization condition (5.13) and* $\tilde{v}>0$. *It admits then one and only one analytic solution tending to zero at infinity.*

To show that this result implies that we have a family of solutions of the Einstein-wave map evolution system which decays to solutions of the VTD equations, we need to verify that for a large enough t we have $\Phi_t = \partial_t \Phi$, $\Phi_a = \partial_a \Phi$ and the like. To show that $\Phi_a = \partial_a \Phi$ we use Eqs. (2.20) together with commutation of partial derivatives to show that:

$$\partial_t(\phi_a - \partial_a \phi) = \partial_a \phi_t - \partial_a \partial_t \phi = 0; \tag{5.28}$$

hence $\phi_a - \partial_a \phi$ is independent of t. As t tends to ∞ it tends to zero because

$$\phi_a - \partial_a \phi = \mathrm{e}^{-\varepsilon_{\phi'} t} u_{\phi_a} - \mathrm{e}^{-\varepsilon_\phi t}(\partial_a u_\phi - \varepsilon_\phi u_\phi). \tag{5.29}$$

It must therefore always be zero. Analogous arguments can be used to show that $\omega_a = \partial_a \omega$ and $\sigma_c^{ab} = \tilde{\nabla}_c \sigma^{ab}$.

6. Constraints

The solution of the evolution system satisfies the full Einstein equations so long as it satisfies also the Einstein constraints, that is

$$C_0 := \Sigma_0^0 \equiv -\frac{1}{2}\{R(g) - k \cdot k + \tau^2 - \mathrm{e}^{-2\lambda} \partial_t \Phi \cdot \partial_t \Phi\} = 0,$$

$$C_a := \mathrm{e}^\lambda \Sigma_a^0 \equiv -\{\nabla_b k_a^b - \partial_a \tau + \mathrm{e}^{-\lambda} \partial_t \Phi \cdot \partial_a \Phi\} = 0.$$

As usual we will rely on the Bianchi identities, here to construct a Fuchsian system satisfied by the constraints. Together with the wave equation satisfied by Φ, the Bianchi identities imply that

$$^{(3)}\nabla_\alpha \Sigma_\beta^\alpha = 0. \tag{6.1}$$

Modulo the evolution equations $^{(3)}R_a^b - \rho_a^b = 0$ that we have solved, with $\rho_a^b \equiv \Phi_a \cdot \Phi^b$, it holds that

$$^{(3)}R - \rho = R_0^0 - \rho_0^0; \tag{6.2}$$

hence

$$\Sigma_0^0 \equiv R_0^0 - \rho_0^0 - \frac{1}{2}\delta_0^0(^{(3)}R - \rho) = \frac{1}{2}\delta_0^0(^{(3)}R - \rho) \tag{6.3}$$

and

$$\Sigma_a^b = -\frac{1}{2}\delta_a^b(^{(3)}R - \rho) = -\delta_a^b \Sigma_0^0. \tag{6.4}$$

We use these equations and the identities

$$\Sigma_a^0 \equiv \mathrm{e}^{-\lambda} C_a, \qquad \Sigma_0^a \equiv -N^2 \Sigma^{a0} \equiv -g^{ab} N^2 \Sigma_b^0 \equiv -\mathrm{e}^\lambda g^{ab} C_b \tag{6.5}$$

together with the expressions for the Christoffel symbols of the metric $^{(3)}g$. We find that Eq. (6.1) can be written in the form

$$\partial_t C_0 - 2\,\mathrm{e}^\lambda \tau C_0 = g^{ab} \nabla_a(\mathrm{e}^\lambda C_b) + g^{ab}\,\mathrm{e}^\lambda \partial_a \lambda C_a \tag{6.6}$$

and (after some simplifications and multiplying by e^λ)

$$\partial_t C_a - e^\lambda \tau C_a = e^\lambda \nabla_a C_0 + 2\, e^\lambda \partial_a \lambda C_0. \tag{6.7}$$

Equivalently, we have

$$\partial_t (e^{2\lambda} C_0) - 2\tilde{v}\, e^{2\lambda} C_0 - 2\, e^\lambda \tau\, e^{2\lambda} C_0 = e^\lambda \sigma^{ab} \nabla_a (e^\lambda C_b) + \sigma^{ab}\, e^\lambda \partial_a \lambda\, e^\lambda C_a \tag{6.8}$$

and

$$\partial_t (e^\lambda C_a) - \tilde{v}\, e^\lambda C_a - e^{2\lambda} \tau C_a = \nabla_a (e^{2\lambda} C_0). \tag{6.9}$$

We see that $e^{2\lambda} C_0$ and $e^\lambda C_a$ satisfy a linear homogeneous system, which admits zero as a solution. This solution is the unique one tending to zero at infinity, so long as the system is Fuchsian.

Lemma 6.1. *The system* (6.8) and (6.9) *is Fuchsian, for a solution of the evolution system, if the VTD solution satisfies* $\hat{C}_0 = 0$ (i.e. $\tilde{v}^2 = \tilde{w}^2$).

Proof. Since the coefficients of Eqs. (6.8) and (6.9) are constructed from solutions of the evolution system we may use the expansions and estimates derived in previous sections. In particular we calculate

$$e^\lambda \tau \equiv \tilde{v} + e^{-\varepsilon_k t} u^a_{k,a}. \tag{6.10}$$

Eq. (6.8) can therefore be written as the following equation of Fuchsian type:

$$\begin{aligned} &\partial_t (e^{2\lambda} C_0) - 4\tilde{v}\, e^{2\lambda} C_0\, e^{2\lambda} C_0 \\ &\quad = e^{-\varepsilon_k t} u^a_{k,a}\, e^{2\lambda} C_0 + e^\lambda \sigma^{ab} \nabla_a (e^\lambda C_b) + \sigma^{ab}\, e^\lambda \partial_a \lambda\, e^\lambda C_a. \end{aligned} \tag{6.11}$$

Eq. (6.9) is not a priori in Fuchsian form for the pair $(e^\lambda C_a, e^{2\lambda} C_0)$ in spite of the identity (6.10). However if we use the identity

$$e^{2\lambda} C_0 \equiv -\frac{1}{2}\{e^{2\lambda} R(g) - e^{2\lambda} k \cdot k + e^{2\lambda} \tau^2 - \partial_t \Phi \cdot \partial_t \Phi\} \tag{6.12}$$

and the property $\hat{C}_0 = 0$ together with the expression for $R(g)$ given in (5.9) we can show that there exists a number $\mu > 0$ and a bounded function $F(x, t)$ such that we have

$$|e^{2\lambda} C_0| \le e^{-\mu t} F(x,t) \quad \text{and} \quad |\partial_a (e^{2\lambda} C_0)| \le e^{-\mu t} F(x,t). \tag{6.13}$$

It follows that (6.9) takes Fuchsian form. □

Theorem 6.2. *A solution of the evolution system satisfies the full Einstein wave map equations if and only if the half polarized asymptotic data satisfies the condition* $\tilde{w} = \tilde{v}$, *and also*

$$\tilde{\Theta} = 1 \quad \textit{and} \quad \tilde{v}\, e^{-\tilde{\lambda} + 2\tilde{\phi}} = \textit{constant}. \tag{6.14}$$

Proof. To complete the proof that $C_0 = C_a = 0$ it suffices to show that $e^{2\lambda} C_0$ and $e^\lambda C_a$ tend to zero at infinity. We have already checked that this is true for $e^{2\lambda} C_0$, as long as $\tilde{w} = \tilde{v}$; i.e. $\hat{C}_0 = 0$.

We now study the asymptotic behavior of $e^{\lambda} C_a$. If we denote by δu the difference between a field u and its VTD value, we calculate (recall that $\lambda = \hat{\lambda}$, $\hat{\sigma} = \tilde{\sigma}$)

$$e^{\lambda}(C_a - \hat{C}_a) \equiv e^{\lambda}\{(\nabla_b - \tilde{\nabla}_b)k_a^b + \tilde{\nabla}_b \delta k_a^b - \partial_a \delta\tau\} + \delta(\Phi_t \cdot \Phi_a) \tag{6.15}$$

with

$$\delta(\Phi_t \cdot \Phi_a) \equiv 2\phi_t \delta\phi_a + 2\hat{\phi}_a \delta\phi_t + \frac{1}{2} e^{-2\phi}\omega_t \delta(e^{-2\phi}\omega_a) + e^{-2\hat{\phi}}\hat{\omega}_a \delta(e^{-2\phi}\omega_t). \tag{6.16}$$

We see that, in the half polarized case, the Fuchsian expansions imply that $e^{\lambda}(C_a - \hat{C}_a)$ tends to zero as t tends to infinity. Using the expressions for $\hat{k}_a^b$ and $\hat{\lambda}$, we see that $e^{\hat{\lambda}}\hat{C}_a$ reads:

$$e^{\hat{\lambda}}\hat{C}_a \equiv \frac{1}{2} e^{\hat{\lambda}} \partial_a(e^{-\hat{\lambda}}\tilde{v}) - \hat{\Phi}_t \cdot \hat{\Phi}_a \equiv \frac{1}{2}(\partial_a \tilde{v} - \tilde{v}\partial_a \tilde{\lambda} + \tilde{v}\partial_a \tilde{v} t) - \hat{\Phi}_t \cdot \hat{\Phi}_a.$$

Using the expressions of $\hat{\lambda}$, $\hat{\Phi}_t$, $\hat{\Phi}_a$ and the half polarization condition, we find after some computation that

$$\hat{\Phi}_t \cdot \hat{\Phi}_a = -\tilde{w}\left\{\cos\theta \partial_a \tilde{\phi} + \frac{1}{2}\tilde{\Theta}^{-1}\partial_a(\tilde{\Theta} - \tilde{w}t)\right\}. \tag{6.17}$$

Thus we find that the terms containing t disappear from $e^{\hat{\lambda}}\hat{C}_a$ if $\tilde{v} = \tilde{w}$ and $\tilde{\Theta} = 1$. It follows that $e^{\hat{\lambda}}\hat{C}_a$ tends to zero as t tends to infinity (recall that $\cos\theta$ tends then to 1) if and only if

$$\frac{1}{2}[\partial_a \tilde{v} - \tilde{v}\partial_a \tilde{\lambda}] - \tilde{v}\partial_a \tilde{\phi} = 0,$$

a condition equivalent to the hypothesis (6.14) of the theorem. □

Remark 6.3. In the half polarized case the VTD solution only satisfies asymptotically the VTD momentum constraint, and only after being multiplied $e^{\hat{\lambda}}$.

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